MATH3705 A — Test 3

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

[**1**] 1. Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \end{cases}$ Let $f_{odd}(x)$ be the 4-periodic **odd** extension of $f(x)$. Which of the following is the expression of $f_{odd}(x)$ when $-1 < x < 0$?

(a) 0 (b) $1+x$ (c) $-1+x$ (d) $1-x$ (e) $-1-x$

Solution: (e)
$f_{odd}(x) = -f(-x) = -(1-(-x)) = -(1+x) = -1-x.$

[**1**] 2. Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \end{cases}$ Assume that, when $x = 11.7$, the Fourier sine series of $f(x)$ converges to B. Find B.

(a) 0.3 (b) -10.7 (c) 11.7 (d) 0 (e) -0.7

Solution: (e)
$B = \frac{f_{odd}(11.7+)+f_{odd}(11.7-)}{2} = f_{odd}(11.7) = f_{odd}(11.7-12) = f_{odd}(-0.3) = -1+0.3 = -0.7.$

[**1**] 3. Let $f(x) = 2x$ on $[0,\ 2]$, let $f(x)$ to be 2-periodic. Determine the value to which the Fourier series of $f(x)$ converges at $x = 47$.

(a) 2 (b) 94 (c) 1 (d) 0.5 (e) 3

Solution: (a).

$L = 1$. At $x = 47$, since $47 = 2(23) + 1$, the series converges to 2.

[1] 4. Consider the differential equation $y'' + y = f(x), \quad -\infty < x < \infty$, where $f(x)$ is 2-periodic and continuous, f' and f'' are piecewise continuous. Assume that the Fourier cosine coefficients and sine coefficients of $f(x)$ are a_n and b_n respectively. Let the 2-periodic solution of the differential equation be

$$y = \frac{c_0}{2} + \sum_{n=1}^{\infty} [c_n \cos(n\pi x) + d_n \sin(n\pi x)].$$

Then $c_2 =$

(a) $\frac{a_2}{1-4\pi^2}$ (b) $\frac{b_2}{1-2\pi^2}$ (c) $\frac{a_2}{1-2\pi^2}$ (d) $\frac{a_2}{1-2\pi}$ (e) $\frac{a_2}{1-4\pi}$

Solution: (a).

$\left(\lambda - \frac{n^2\pi^2}{L^2}\right) c_n = a_n$

[10] 5. [2 + 4 + 4 points] Let

$$f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in [0, \pi). \end{cases}$$

and let $f(x)$ be 2π-periodic. Find the Fourier coefficients a_0, a_n, b_n.

Solution:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_0^{\pi} 1dx\right) = 1, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_0^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^{\pi} = 0, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_0^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^{\pi} \\
&= \frac{1}{n\pi}(1-(-1)^n) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}
\end{aligned}$$

[6] 6. [2 + 4 points] Let $f(x) = \begin{cases} 1-x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \end{cases}$ Find a_0 and a_n of the Fourier cosine series.

Solution: Solution:

$$\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\int_0^2 f(x)\,dx \\
&= \int_0^1 (1-x)\,dx = \frac{1}{2}.
\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{n\pi x}{2})\,dx \\
&= \int_0^1 (1-x)\cos(\frac{n\pi x}{2})\,dx \\
&= \left[\frac{2}{n\pi}(1-x)\sin(\frac{n\pi x}{2}) - \frac{4}{n^2\pi^2}\cos(\frac{n\pi x}{2})\right]_0^1 \\
&= \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos(\frac{n\pi}{2}).
\end{aligned}
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